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Complete solutions for a model with chiral dynamics

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Abstract. A mechanical analogue for the chiral SU(2) × SU(2) invariant field theory for massless pions has as its classical lagrangian $L = \frac{1}{2}\dot{q}^i g_{ij} \dot{q}^j$. The coordinates q^i parametrize a manifold (in fact a three-sphere) on which $\mathrm{d}q^i g_{ij}\,\mathrm{d}q^j$ is the SU(2) × SU(2) invariant metric. The classical equations of motion are for uniform geodesic motion, and in the corresponding quantum mechanical case the Schrödinger equation is the Laplace–Beltrami–Poisson equation. Both sets of equations are solved completely with attention paid to the conserved quantities, that is, the generators of the invariance group SU(2) × SU(2). The passage to the limiting case, in which the manifold becomes flat, is also described.

1. Introduction

We have recently (Charap 1973) considered a simple model analogue of the chiral invariant dynamics of zero-mass mesons. In this model, instead of considering a multiplet of fields, with the appropriate gradient self-couplings of a chiral invariant theory, we work with a system having only a finite number of degrees of freedom. Thus in the case of SU(2) × SU(2), instead of the isotopic triplet of pion fields $\boldsymbol{\phi}(\vec{x}, t)$, we consider an isotopic triplet of variables $\boldsymbol{q}(t)$. It is important to emphasize that there is no $\vec{x}$ dependence in the model whatever; and of course the time t enters parametrically into the theory. The classical system has a lagrangian

$$L = \tfrac{1}{2}\dot{q}^i g_{ij}(q)\dot{q}^j. \tag{1.1}$$

In this expression we have introduced the velocities

$$\dot{q}^i = \frac{\mathrm{d}q^i}{\mathrm{d}t}. \tag{1.2}$$

where of course q^i $(i = 1, 2, 3)$ are the components of the coordinate triplet $\boldsymbol{q}$. The matrix g_{ij} plays the role of a metric on a manifold, the points of which have coordinates $\boldsymbol{q}$. The form of g_{ij}, which is chosen to make the lagrangian invariant, depends on the transformation of $\boldsymbol{q}$ under the action of the group SU(2) × SU(2), which is given an action point-wise on the manifold. With the usual separation of the generators of the group into $\boldsymbol{V}$ (the triplet of generators of the parity-conserving diagonal isospin subgroup) and $\boldsymbol{A}$ (the pure chiral generators) we have the algebra

$$\begin{aligned}
[V^i, V^j] &= \mathrm{i}\epsilon^{ij}{}_k V^k, \\
[V^i, A^j] &= \mathrm{i}\epsilon^{ij}{}_k A^k, \\
[A^i, A^j] &= \mathrm{i}\epsilon^{ij}{}_k V^k;
\end{aligned} \tag{1.3}$$

and the transformation law

$$[V^i, q^j] = \mathrm{i}\epsilon^{ij}{}_k q^k, \tag{1.4}$$

$$[A^i, q^j] = \mathrm{i}f^{ij}(q). \tag{1.5}$$

There is an undetermined function $f(q^2)$, with $q^2 = q^i q^j \delta_{ij}$, which, once specified, fixes both f^{ij} and g_{ij}. We shall remind the reader of the relevant formulae below. Although equations (1.3)–(1.5) have been written with commutator brackets, that is, in a form appropriate for a quantum-mechanical treatment, exactly similar equations also hold in classical mechanics, with the usual substitution

$$\{\text{Poisson bracket}\} \to -\mathrm{i}[\text{commutator}]. \tag{1.6}$$

The equation of motion for the classical system with lagrangian (1.1) is

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0, \tag{1.7}$$

with the Christoffel symbol defined as usual

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}). \tag{1.8}$$

We adopt the notation g^{il} for the elements of the matrix inverse to the metric g_{jk}; also $g_{lj,k}$ denotes $\partial g_{lj}/\partial q^k \equiv \partial_k g_{lj}$, etc.

If we write g for the determinant of the metric

$$g \equiv \det\|g_{ij}\|, \tag{1.9}$$

the Laplace–Beltrami operator on the manifold parametrized by $\boldsymbol{q}$ is

$$\Delta_2 = g^{-1/2}\partial_i g^{1/2} g^{ij}\partial_j. \tag{1.10}$$

Then in a quantum-mechanical treatment of the analogous system, still with chiral invariance, we may set up a Schrödinger representation with $\boldsymbol{q}$ diagonal and wave-functions $\psi(\boldsymbol{q}, t)$ normalized so that

$$\langle\psi|\psi\rangle \equiv \int |\psi(q,t)|^2 g^{1/2}\,\mathrm{d}^3 q = 1. \tag{1.11}$$

To preserve this group-invariant normalization, the time-dependent Schrödinger equation is

$$\mathrm{i}\mathrm{D}_t\psi = H\psi, \tag{1.12}$$

with the conservative time-derivative D_t given by

$$\mathrm{D}_t = g^{-1/4}\partial/\partial t\, g^{1/4}, \tag{1.13}$$

and the hamiltonian H given by

$$H = -\tfrac{1}{2}\Delta_2. \tag{1.14}$$

If, as in the case of present interest, g has no explicit time dependence, $\mathrm{D}_t = \partial/\partial t$, and the Schrödinger equation is

$$\mathrm{i}\partial\psi/\partial t = H\psi. \tag{1.15}$$

The object of this paper is to give complete, exact solutions of the classical and quantum-mechanical equations of motion, namely, of the equations (1.7) and (1.12). Before we do this we shall, in the next section, give a brief summary of some well known expressions for the metric and related quantities in a few specific parametrizations. Then in § 3 we discuss and solve the classical equations of motion, and in § 4 do the same for the quantum-mechanical case.

2. The metric and related quantities

As remarked in the previous section, the chiral transformations of the coordinates $\boldsymbol{q}$, as given by (1.5), leave unspecified a single function $f(q^2)$ (see Weinberg 1968). To be explicit, if for a given function $f(q^2)$ we define $h(q^2)$ by

$$h(q^2) = \frac{(f(q^2))^2 + q^2}{f(q^2) - 2q^2 f'(q^2)}, \tag{2.1}$$

then a possible form for the quantity f^{ij} in (1.5) is

$$f^{ij} = fP^{ij} + hQ^{ij}, \tag{2.2}$$

where we have found it convenient to introduce the projections defined by

$$P^{ij} \equiv \delta^{ij} - Q^{ij}, \tag{2.3a}$$

$$Q^{ij} \equiv q^i q^j / q^2. \tag{2.3b}$$

A change of parametrization of the manifold consistent with leaving invariant the form of (1.4) is

$$\boldsymbol{q} \to \boldsymbol{q}^* = \boldsymbol{q}\phi(q^2). \tag{2.4}$$

The resulting transformation under the axial generators is then given by an expression like (1.5), but with f^{ij} replaced by what results from (2.2) by writing $\boldsymbol{q}^*$ everywhere instead of $\boldsymbol{q}$ and changing $f(q^2)$ to $f^*(q^{*2})$ defined through

$$f^*(q^{*2}) = f^*\{q^2(\phi(q^2))^2\} = f(q^2)\phi(q^2). \tag{2.5}$$

Expressed in terms of the function $f(q^2)$, the arbitrariness of which is evidenced through (2.5), the metric g_{ij} which makes invariant the lagrangian (1.1) is†

$$g_{ij} = f_\pi^2\{(f^2+q^2)^{-1}P_{ij} + h^{-2}Q_{ij}\}. \tag{2.6}$$

The quantity f_π is a constant, usually chosen so that $g_{ij} = \delta_{ij}$ when $q = 0$, in which case it is clear that $f_\pi \equiv f(0)$. We also have directly

$$g^{ij} = f_\pi^{-2}\{(f^2+q^2)P^{ij} + h^2 Q^{ij}\}, \tag{2.7}$$

and

$$g = \{f_\pi^{-3}(f^2+q^2)h\}^{-2}. \tag{2.8}$$

2.1. *σ model parametrization*

One very common choice (eg Bardeen and Lee 1969) for $f(q^2)$ is simply‡

$$f_\sigma(q_\sigma^2) = (f_\pi^2 - q_\sigma^2)^{1/2}. \tag{2.9}$$

† No significance should be attached to the difference between upper and lower indices on the projections P and Q.

‡ The suffix σ is introduced to distinguish this parametrization.

Evidently with this choice, since

$$q_\sigma^2 + f_\sigma^2 = f_\pi^2, \tag{2.10}$$

the three coordinates q_σ^i together with f_σ may be seen to give the coordinates on a three-sphere of radius f_π of a point $(\boldsymbol{q}_\sigma, f)$ in a four-dimensional space. Indeed the manifold parametrized by $\boldsymbol{q}_\sigma$ is just this three-sphere S_3. If we introduce the angle θ (see figure 1) by

$$q_\sigma = f_\pi \sin\theta, \tag{2.11}$$

we evidently have

$$f_\sigma = f_\pi \cos\theta, \tag{2.12}$$

and also

$$h_\sigma = f_\pi \cos\theta, \tag{2.13}$$

$$f_\sigma^{ij} = f_\pi \cos\theta\delta^{ij}, \tag{2.14}$$

$$g_{\sigma ij} = P_{\sigma ij} + \sec^2\theta Q_{\sigma ij}, \tag{2.15}$$

$$g_\sigma = \sec^2\theta. \tag{2.16}$$

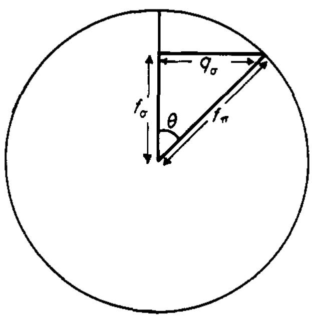

Figure 1. The σ model parametrization.

2.2. Weinberg's parametrization

An alternative parametrization (Weinberg 1968) may be obtained by stereographic projection onto the tangent hyperplane at $\boldsymbol{q} = 0$ from the opposite pole of S_3 (see figure 2). Thus we may write

$$q_W = 2f_\pi \tan\tfrac{1}{2}\theta, \tag{2.17}$$

from which follows

$$f_W = f_\pi(1 - \tan^2\tfrac{1}{2}\theta), \tag{2.18}$$

$$h_W = f_\pi(1 + \tan^2\tfrac{1}{2}\theta), \tag{2.19}$$

$$g_{Wij} = \cos^4\tfrac{1}{2}\theta\delta_{ij}, \tag{2.20}$$

$$g_W = \cos^{12}\tfrac{1}{2}\theta. \tag{2.21}$$

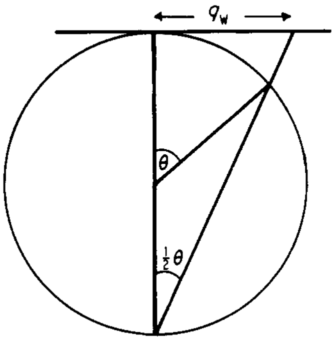

Figure 2. Weinberg's parametrization.

2.3. *Normal coordinates*

Another obviously useful parametrization (cf, eg, Callan *et al* 1969, see also Coleman *et al* 1969, and Isham 1969) is given by

$$q_N = f_\pi \theta, \tag{2.22}$$

for which we have

$$f_N = f_\pi \theta \cot \theta, \tag{2.23}$$

$$h_N = f_\pi, \tag{2.24}$$

$$g_{Nij} = P_{Nij}(\sin \theta/\theta)^2 + Q_{Nij}, \tag{2.25}$$

$$g_N = (\sin \theta/\theta)^4. \tag{2.26}$$

2.4. *Tangential parametrization*

The choice of parametrization we shall most often use is another kind of stereographic projection, this time from the *centre* of S_3 onto the tangent hyperplane at $\boldsymbol{q} = 0$ (see figure 3). We will use symbols without a distinguishing suffix for this parametrization. Accordingly we have

$$q = f_\pi \tan \theta, \tag{2.27}$$

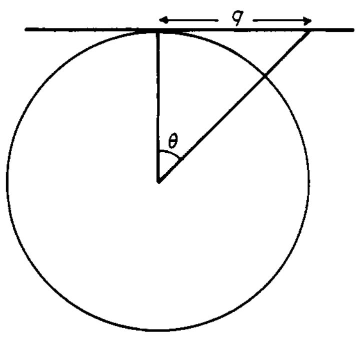

Figure 3. Tangential parametrization.

$$f = f_\pi, \tag{2.28}$$

$$h = f_\pi \sec^2\theta, \tag{2.29}$$

$$g_{ij} = \cos^2\theta P_{ij} + \cos^4\theta Q_{ij}, \tag{2.30}$$

$$g = \cos^8\theta. \tag{2.31}$$

This parametrization leads to a very simple expression for the Christoffel symbol, namely

$$\Gamma^l_{ij} = -f_\pi^{-2}\cos^2\theta(\delta_{ik}\delta^l_j + \delta_{jk}\delta^l_i)q^k. \tag{2.32}$$

3. The classical equations of motion

The equation of motion (1.7) is that for uniform motion along the geodesics of the manifold with coordinates $\boldsymbol{q}$ and metric g_{ij}. Thus in the present case it is uniform motion on the geodesics of S_3, and these are of course the great circles of the hypersphere. The advantage of the tangential parametrization is now evident, since every great circle is mapped by the stereographic projection from the centre of the hypersphere into a *straight line* in the tangent hyperplane. Of course the motion is no longer uniform along the straight line projection. Furthermore as the coordinate varies once around the great circle, the image point traverses the straight line twice.

Using (2.32), the equations of motion become

$$\ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}}\, \mathrm{d}(ln \sec^2\theta)/\mathrm{d}t, \tag{3.1}$$

together with

$$\boldsymbol{q}^2 = f_\pi^2 \tan^2\theta. \tag{3.2}$$

Let us define $\boldsymbol{a}$ by

$$\boldsymbol{a} = -f_\pi^{-1}\cos^2\theta\dot{\boldsymbol{q}}. \tag{3.3}$$

Then it is an immediate consequence of (3.1) that the isotriplet $\boldsymbol{a}$ is a constant of the motion,

$$\dot{\boldsymbol{a}} = 0. \tag{3.4}$$

We make the ansatz

$$\boldsymbol{q} = f_\pi(\boldsymbol{b} - \boldsymbol{a}h), \tag{3.5}$$

with

$$\boldsymbol{b}\,.\,\boldsymbol{a} \equiv \delta_{ij}b^i a^j = 0. \tag{3.6}$$

Then from

$$\dot{\boldsymbol{q}} = f_\pi(\dot{\boldsymbol{b}} - \dot{\boldsymbol{a}}h - \boldsymbol{a}\dot{h}) = f_\pi(\dot{\boldsymbol{b}} - \boldsymbol{a}\dot{h}), \tag{3.7}$$

together with (3.3) and (3.6), it follows that

$$\dot{\boldsymbol{b}} = 0. \tag{3.8}$$

It is convenient to introduce instead of $\boldsymbol{b}$ another constant isovector $\boldsymbol{v}$, defined so that

$$\dot{\boldsymbol{v}} = 0, \qquad \boldsymbol{v}\,.\,\boldsymbol{a} = 0, \tag{3.9}$$

$$\boldsymbol{b} = \boldsymbol{v}\times\boldsymbol{a}/a^2, \tag{3.10}$$

where in the usual way the 'vector product' is defined so that

$$(\boldsymbol{v}\times\boldsymbol{a})^i = \epsilon^i{}_{jk}v^j a^k. \tag{3.11}$$

Then we have

$$\boldsymbol{v} = f_\pi^{-1}\boldsymbol{a}\times\boldsymbol{q}. \tag{3.12}$$

Using

$$q^2 = f_\pi^2(h^2a^2+v^2/a^2), \tag{3.13}$$

and introducing ω through

$$\omega^2 = v^2+a^2, \tag{3.14}$$

we have

$$\sec^2\theta = (q^2+f_\pi^2)/f_\pi^2 = h^2a^2+\omega^2/a^2, \tag{3.15}$$

and

$$\dot{\boldsymbol{q}} = -f_\pi \boldsymbol{a}\dot{h}. \tag{3.16}$$

Comparing with (3.3) we obtain

$$\dot{h} = h^2a^2+\omega^2/a^2, \tag{3.17}$$

so that

$$a^2h = \omega\tan\{\omega(t-t_0)\}, \tag{3.18}$$

where t_0 is a constant of integration. This means that the complete solution of the equations of motion (3.1) is given by

$$\boldsymbol{q} = f_\pi[\boldsymbol{v}\times\boldsymbol{a}-\omega\boldsymbol{a}\tan\{\omega(t-t_0)\}]/a^2. \tag{3.19}$$

The significance of the constant isovectors $\boldsymbol{v}$ and $\boldsymbol{a}$ may be seen by recognizing that the generators $\boldsymbol{V}$ and $\boldsymbol{A}$ introduced in (1.3) are given explicitly in terms of the coordinates $\boldsymbol{q}$ and the velocities $\dot{\boldsymbol{q}}$ by

$$V^i = -\epsilon^{ij}{}_k q^k p_j, \tag{3.20}$$

$$A^i = -f^{ij}p_j, \tag{3.21}$$

with

$$p_j = g_{jk}\dot{q}^k. \tag{3.22}$$

This means that we have, with the present coordinates

$$\boldsymbol{A} = -f_\pi^3(f_\pi^2+q^2)^{-1}\dot{\boldsymbol{q}}, \tag{3.23}$$

$$\boldsymbol{V} = f_\pi^{-1}\boldsymbol{A}\times\boldsymbol{q}. \tag{3.24}$$

If we evaluate these expressions for the motion given by (3.19) we obtain

$$\boldsymbol{A} = f_\pi^2\boldsymbol{a} \qquad \boldsymbol{V} = f_\pi^2\boldsymbol{v}. \tag{3.25}$$

So we may rewrite (3.19) as

$$\boldsymbol{q} = f_\pi\{\boldsymbol{V}\times\boldsymbol{A}-f_\pi^2\omega\boldsymbol{A}\tan\{\omega(t-t_0)\}]/A^2, \tag{3.26}$$

with

$$f_\pi^4\omega^2 = V^2+A^2. \tag{3.27}$$

The hamiltonian for the motion may be written as

$$H = (V^2 + A^2)/2f_\pi^2, \tag{3.28}$$

so that the energy E of the motion is given by

$$E = \tfrac{1}{2} f_\pi^2 \omega^2. \tag{3.29}$$

As previously indicated, the trajectories in the tangent hyperplane are straight lines. It is amusing to consider the limit $f_\pi \to \infty$ taken in conjunction with

$$V \to J = \text{constant} \tag{3.30a}$$

$$\boldsymbol{u} \equiv -\boldsymbol{A}/f_\pi \to \text{constant.} \tag{3.30b}$$

For then we find

$$E \to \tfrac{1}{2} u^2, \tag{3.31}$$

and in the limit the trajectory becomes

$$\boldsymbol{q} \simeq \boldsymbol{u}(t - t_0) + \boldsymbol{J} \times \boldsymbol{u}/u^2, \tag{3.32}$$

which is of course that of a particle moving freely with uniform velocity $\boldsymbol{u}$ and angular momentum $\boldsymbol{J}$; the energy is appropriate for such a motion because the choice of scale, so that $g_{ij} = \delta_{ij}$ when $q = 0$, ensures that in the limit the lagrangian becomes just $\frac{1}{2}\dot{\boldsymbol{q}}^2$, appropriate to a free particle of unit mass.

4. The quantum-mechanical equation of motion

It follows from (1.12) that the general wavefunction is a linear superposition of stationary state wavefunctions, which in the usual way satisfy the time-independent Schrödinger equation

$$H\psi = E\psi, \tag{4.1}$$

and have time dependence

$$\psi = \psi(t = 0)\exp(-\mathrm{i}Et). \tag{4.2}$$

Accordingly we turn our attention to the solution of (4.1) which, with the form (1.14) for the hamiltonian, becomes

$$\Delta_2 \psi + 2E\psi = 0. \tag{4.3}$$

Guided by the considerations of the previous section, we will write $\boldsymbol{q}$ in terms of polar coordinates, thus

$$\boldsymbol{q} = (q, \alpha, \beta). \tag{4.4}$$

Introducing $\boldsymbol{L}$ by

$$\boldsymbol{L} = -\mathrm{i}\boldsymbol{\nabla} \times \boldsymbol{q}, \tag{4.5}$$

$$\nabla_i \equiv \partial/\partial q^i, \tag{4.6}$$

we have, as usual

$$q^2 \nabla_i P^{ij} \nabla_j = -\boldsymbol{L}^2 \tag{4.7}$$

$$\boldsymbol{q} \cdot \boldsymbol{\nabla} = q\partial/\partial q, \tag{4.8}$$

and, for $q \neq 0$,

$$\nabla^2 = \left(\frac{1}{q}\frac{\partial}{\partial q}q\right)^2 - \frac{\boldsymbol{L}^2}{q^2}. \tag{4.9}$$

Furthermore, it follows directly from the definition (1.10) of the Laplace–Beltrami operator, and the expression (2.7) for g^{ij}, that we have

$$\Delta_2 = f_\pi^{-2}\left((f^2+q^2)\nabla_i P^{ij}\nabla_j + h^2\nabla_i Q^{ij}\nabla_j + \frac{g^{-1/2}}{q}\frac{\mathrm{d}}{\mathrm{d}q}(h^2 g^{1/2})\boldsymbol{q}\,.\,\nabla\right). \tag{4.10}$$

Using (2.3*a*) and (4.7) to (4.9), this results in the form, valid for $q \neq 0$,

$$\Delta_2 = f_\pi^{-2}\left\{h^2\left(\frac{1}{q}\frac{\partial}{\partial q}q\right)^2 - \frac{f^2+q^2}{q^2}\boldsymbol{L}^2 + g^{-1/2}\frac{\mathrm{d}}{\mathrm{d}q}(h^2 g^{1/2})\frac{\partial}{\partial q}\right\}. \tag{4.11}$$

To include also the point $q = 0$, the square integrability condition

$$\int |\psi|^2 g^{1/2}\,\mathrm{d}^3 q = 1, \tag{4.12}$$

must be supplemented by the condition

$$\lim_{q\to 0} q\psi(\boldsymbol{q}) = 0. \tag{4.13}$$

This condition is of course already familiar from the analogous situation which arises on the usual separation by polar coordinates of the Schrödinger equation for a single particle in three dimensions.

We now recognize that the operator L^2 in (4.11) acts only on the variables α, β, so that if we write

$$\psi_l(\boldsymbol{q}) = \phi_l(q)Y_{lm}(\alpha, \beta), \tag{4.14}$$

we have

$$\boldsymbol{L}^2\psi_l(\boldsymbol{q}) = l(l+1)\psi_l(\boldsymbol{q}), \tag{4.15}$$

in the usual way. Furthermore, since any wavefunction can be expanded as a superposition of solutions like (4.14), we are left with the problem of solving

$$f_\pi^{-2}\left\{h^2\left(\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}q}q\right)^2 - \frac{f^2+q^2}{q^2}l(l+1) + g^{-1/2}\frac{\mathrm{d}}{\mathrm{d}q}(h^2 g^{1/2})\frac{\mathrm{d}}{\mathrm{d}q}\right\}\phi_l(q) = E\phi_l(q), \tag{4.16}$$

subject to the conditions

$$\lim_{q\to 0} q\phi_l(q) = 0, \tag{4.17}$$

and

$$4\pi\int |\phi_l(q)|^2 g^{1/2}(q) q^2\,\mathrm{d}q = 1. \tag{4.18}$$

The different parametrizations discussed in § 2 correspond to different ways of expressing the variable q in terms of, let us say, the angle θ. The invariant choice of normalization (4.18) means that we always find

$$g^{1/2}(q)q^2\,\mathrm{d}q = f_\pi^3 \sin^2\theta\,\mathrm{d}\theta. \tag{4.19}$$

If we go, for example, to the particular parametrization used in the previous section, so that once again

$$q = f_\pi \tan\theta, \tag{4.20}$$

which has the advantage that $h^2 g^{1/2} = f_\pi^2$ so that the last term on the left-hand side of (4.16) is absent, we can obtain the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} - \frac{l(l+1)}{\sin^2\theta} + e + 1\right) v(\theta) = 0, \tag{4.21}$$

where we have defined

$$v(\theta) = \sin\theta\phi_l, \tag{4.22}$$

and

$$e = 2Ef_\pi^2. \tag{4.23}$$

The square integrability condition is now

$$4\pi f_\pi^3 \int_0^{2\pi} |v(\theta)|^2 \,\mathrm{d}\theta = 1, \tag{4.24}$$

and the regularity condition at $q = 0$ becomes

$$v(\theta) = 0 \qquad \text{for } \theta = 0, \pi. \tag{4.25}$$

Introduce z and w by

$$z = \sin^2\theta, \tag{4.26}$$

$$w = (\sin\theta)^{-l-1} v, \tag{4.27}$$

and we obtain the hypergeometric equation

$$z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}w + \{(l+\tfrac{3}{2}) - (l+2)z\}\frac{\mathrm{d}}{\mathrm{d}z}w - \tfrac{1}{4}\{(l+1)^2 - (e+1)\}w = 0. \tag{4.28}$$

With the identification of parameters

$$\alpha = \tfrac{1}{2}\{l+1+\surd(e+1)\}, \qquad \beta = \tfrac{1}{2}\{l+1-\surd(e+1)\}, \qquad \gamma = l+\tfrac{3}{2}; \tag{4.29}$$

we obtain for v the general solution

$$v = Az^{1/2(l+1)}F(\alpha,\beta;\gamma;z) + Bz^{-1/2l}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z). \tag{4.30}$$

The condition (4.25) applied for $\theta = 0$ implies, for each value of l, that $B = 0$. In order to apply the condition at $\theta = \pi$, it is necessary to increase θ through $\tfrac{1}{2}\pi$, so that the variable z increases to $z = 1$ and then decreases again to zero. As this happens it is important to go around the branch point (at $z = 1$) of the function v (see figure 4). The

Figure 4. The branch cuts of the function v. As θ increases from zero to π, the point z follows a contour as indicated, going round the branchpoint at $z = 1$ as θ goes through $\tfrac{1}{2}\pi$ to emerge onto the second sheet of v (where the contour is shown as a broken line).

solution (4.30), with B set to zero, may be written as the sum of two contributions,

$$v = v_e + v_0, \tag{4.31}$$

where v_e is regular at $z = 1$, and v_0 has a square root type branchpoint at $z = 1$. The effect of going round the branchpoint at $z = 1$ is thus to leave v_e unaltered but to change the sign of v_0. We have ensured that the condition (4.25) is satisfied at $\theta = 0$, which means that the combination (4.31) vanishes at $z = 0$. To satisfy (4.25) also at $\theta = \pi$ means therefore that $v_e - v_0$ *also* vanishes at $z = 0$. But any two functions v which satisfy (4.21) and vanish at $z = 0$ must be proportional, so that either v_e or v_0 must vanish *identically*. It is easy to confirm that we have

$$\begin{aligned} v_e &= A\{\Gamma(\alpha+\tfrac{1}{2})\Gamma(\beta+\tfrac{1}{2})\}^{-1} f_e, \\ v_0 &= A\{\Gamma(\alpha)\Gamma(\beta)\}^{-1} f_0, \end{aligned} \tag{4.32}$$

where f_e and f_0 do *not* vanish identically. Hence we require that one of $\alpha, \beta, \alpha+\frac{1}{2}, \beta+\frac{1}{2}$ is a negative integer or zero. This will be true if and only if

$$e+1 = (k+l+1)^2 \qquad \text{for } k = 0, 1, 2, 3, \ldots . \tag{4.33}$$

The solution then becomes

$$v = A(\sin\theta)^{l+1} F(l+1+\tfrac{1}{2}k, -\tfrac{1}{2}k; l+\tfrac{3}{2}; \sin^2\theta), \tag{4.34}$$

which can also be expressed in terms of the Gegenbauer polynomial C_k^{l+1} as

$$v = N(\sin\theta)^{l+1} C_k^{l+1}(\cos\theta). \tag{4.35}$$

The normalization constant required for conformity with (4.24) is

$$N = \pi^{-1} 2^{l-1} l! \{(k+l+1)k!\}^{1/2} \{f_\pi^3 (k+2l+1)!\}^{-1/2}. \tag{4.36}$$

What we have found is that the energy eigenvalues are given by

$$E = \tfrac{1}{2} f_\pi^{-2} 4j(j+1), \tag{4.37}$$

with

$$j = \tfrac{1}{2}(k+l) = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots . \tag{4.38}$$

This is exactly what was to be expected from the observation which follows from (3.28), namely that we may define operators $\boldsymbol{V}_\pm$ by

$$\boldsymbol{V}_\pm = \tfrac{1}{2}(\boldsymbol{V} \pm \boldsymbol{A}), \tag{4.39}$$

which themselves satisfy 'angular momentum' commutation relations, and since $\boldsymbol{V} . \boldsymbol{A} = 0$, this enables us to write

$$H = \tfrac{1}{2} f_\pi^{-2} 4 \boldsymbol{V}_+^2 = \tfrac{1}{2} f^{-2} 4 \boldsymbol{V}_-^2. \tag{4.40}$$

The eigenvalue relations (4.37), (4.38) follow directly. It should be emphasized that the possible values of l are *integers* because of the structure of (4.5). However, no such structure is valid for $\boldsymbol{V}_\pm$, and half-integer eigenvalues for j are permissible.

It is once again interesting to examine the limit $f_\pi \to \infty$, with $K = kf_\pi^{-1}$ and $r = \theta f_\pi$ approaching finite values. A quadratic transform of the hypergeometric function in (4.34) gives

$$F(l+1+\tfrac{1}{2}k, -\tfrac{1}{2}k; l+\tfrac{3}{2}; \sin^2\theta) = e^{ik\theta} F(-k, l+1; 2l+2; 2i\sin\theta\, e^{-i\theta}). \tag{4.41}$$

Under the above limiting process this gives, by confluence of the hypergeometric function

$$e^{iKr}{}_1F_1(l+1\,;2l+2\,;\,-2iKr) = 2\Gamma(l+\tfrac{3}{2})(\tfrac{1}{2}Kr)^{-l}\pi^{-1/2}j_l(Kr). \tag{4.42}$$

We also have

$$\begin{aligned} A(\sin\theta)^{l+1} &= \Gamma(2l+k+2)\{\Gamma(2l+2)\Gamma(k+1)\}^{-1}N(\sin\theta)^{l+1} \\ &= \frac{(\sin\theta)^{l+1}}{2^{l+2}\Gamma(l+\frac{3}{2})}\left(\frac{\Gamma(2l+k+2)(k+l+1)}{\pi f_\pi^3\Gamma(k+1)}\right)^{1/2} \\ &\sim \tfrac{1}{2}(\tfrac{1}{2}Kr)^{l+1}(\pi f_\pi^3)^{-1/2}\{\Gamma(l+\tfrac{3}{2})\}^{-1}. \end{aligned} \tag{4.43}$$

Hence in the limit we have

$$v \sim (2\pi)^{-1}f_\pi^{-3/2}(Kr)j_l(Kr), \tag{4.44}$$

which is just as was to be expected the regular solution to the equation

$$\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+K^2\right)u(r) = 0; \tag{4.45}$$

this equation, which may be obtained from (4.21) by going to the limit, is of course the reduced radial wave equation for a wavefunction of angular momentum l in a flat space.

References

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